Answers to Selected Exercises in ***Principles of Mathematical Analysis***[*]

Solutions by

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CHAPTER 2. BASIC TOPOLOGY

1. *Prove that the empty set is a subset of every set.*

Proof. Let S be an arbitrary set. Then every element of $\emptyset$ is an element of S. So $\emptyset \subseteq S$. ∎

2. *A complex number z is said to be algebraic if there are integers $a_0, \ldots, a_n$, not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. Hint: For every positive integer N, there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N$$

Proof. Let A denote the set of all algebraic numbers and partition A as follows: For each $z \in A$, calculate the positive integer N that corresponds to its equation and place it in a set E_N. So

$$A = \bigcup_{N \in \mathbf{N}} E_N,$$

where each E_N is finite. Then apply the corollary of Theorem 2.12 to find that A is countable. ∎

3. *Prove that there exist real numbers which are not algebraic.*

Proof. Let A denote the set of all algebraic numbers and suppose, towards a contradiction, that all real numbers are algebraic. Then $\mathbf{R} \subseteq A$. But the set of real numbers is uncountable and we know from Problem 2 that A is countable. The contradiction completes the proof. ∎

4. *Is the set of all irrational real numbers countable?*

The answer is no.

Proof. If $\mathbf{R} \setminus \mathbf{Q}$ is countable, then $\mathbf{R} = \mathbf{Q} \cup (\mathbf{R} \setminus \mathbf{Q})$, is countable, which we know to be false. ∎

5. *Construct a bounded set of real numbers with exactly three limit points.*

$\{\frac{1}{n} : n \in \mathbf{N}\} \cup \{\frac{1}{n} + 2 : n \in \mathbf{N}\} \cup \{\frac{1}{n} + 4 : n \in \mathbf{N}\}$ has limit points 0, 2, and 4. It is bounded above by 6 and below by 0.

CHAPTER 11. THE LEBESGUE THEORY

1. *If $f \geq 0$ and $\int_E f\,d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. Hint: Let E_n be the subset of E on which $f(x) > 1/n$. Write $A = \bigcup_{n=1}^{\infty} E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.*

Proof. First we claim that $\mu(E_n) = 0$ for every n. Suppose, towards a contradiction, that $\mu(E_n) > 0$ for some n. Then since $f(x) > 1/n$ for all $x \in E_n$, we find that

$$\int_E f\,d\mu \geq \int_{E_n} f\,d\mu > \frac{\mu(E_n)}{n} > 0,$$

contradicting our hypothesis. Now since $\mu(E_n) = 0$ for all n, $\mu(A) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$. So $\mu(A) = 0$. Now we write

$$\int_E f\,d\mu = \int_{E \setminus A} f\,d\mu + \int_A f\,d\mu,$$

and f is equal to 0 on $E \setminus A$ by our construction of A. ∎

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[*] Walter Rudin. 1986. *Principles of Mathematical Analysis, 3rd ed.* McGraw-Hill, Inc., USA.