Answers to Selected Exercises in *Modern Algebra*[*]

Solutions by

Marcel K. Goh

CHAPTER I: NUMBERS AND SETS

2. Mappings. Cardinality

1. *For an arbitrary set A, prove that $A \sim A$.*

Proof. For each element $a \in A$, let $\phi(a) = a$. It is easy to see that this is a one-to-one correspondence. ∎

2. *Given sets A and B, prove that $A \sim B$ implies $B \sim A$.*

Proof. Since $A \sim B$, there exists a one-to-one correspondence ϕ from A onto B. Then ϕ^{-1} is a one-to-one correspondence from B onto A. ∎

3. *For sets A, B, and C, prove that if $A \sim B$ and $B \sim C$, then $A \sim C$.*

Proof. We have the existence of biunique mappings $\phi : A \to B$ and $\psi : B \to C$. Then $\psi\phi$ is a one-to-one correspondence from A to C (with $\psi^{-1}\phi^{-1}$ as its inverse). ∎

3. The Number Sequence

1. *Let a property E be true, first for $n = 3$, and second for $n + 1$ whenever it is true for $n \geq 3$. Prove that E is true for all numbers ≥ 3.*

Proof. For a number k, let F denote the property "E is true for $k+2$". Then E is true for all numbers $n \geq 3$ if and only if F is true for all natural numbers k. We find that F is true for $k = 1$, since E is true for $n = 3$. Then from the second statement about the property E, we can derive that also F is true for $k + 1$ whenever it is true for $k \geq 1$. So by the principle of complete induction, we have F true for all natural numbers. ∎

3. *The same as Ex. 1 with the number 3 replaced by 0.*

Proof. For a natural number k, let F denote the property "E is true for $k - 1$". Then proceed as in the solution to Exercise 1. ∎

CHAPTER II: GROUPS

6. The Group Concept

1. *The Euclidean motions of space combined with reflections (i.e. those transformations that preserve all distances between pairs of points) form an infinite non-abelian group.*

Proof. We work in 2-dimensional space, but most of the work generalises to higher dimensions. Any Euclidean motion M can be described as one of the following:

a) a translation t_{PQ} that takes a point P to a point Q;
b) a rotation $s_P(\theta)$ of θ radians about the point P;
c) a reflection r_H where H is a hyperplane (line).

(Note that the same motion may be described in multiple ways. For example, if PQ and ST are parallel line-segments with the same length, then $t_{PQ} = t_{ST}$.) The motion t_{PP} that fixes every point satisfies

[*] B. L. van der Waerden, *Modern Algebra*, translated by Fred Blum, New York: Ungar, 1949.

the requirements of an identity element. Associativity follows from the fact that, for any point X in Euclidean space, multiplication equates to composing motions. Thus $(M_1 M_2)M_3(X) = M_1\big(M_2(M_3(X))\big) = M_1(M_2 M_3)(X)$; since the two motions act identically on every point in the space, they are the same transformation. To see that every transformation has an inverse, we need only note that $(t_{PQ})^{-1} = t_{QP}$ for all points P and Q; $(s_P(\theta))^{-1} = s_P(-\theta)$ for all points P and choices of θ; and $(r_H)^{-1} = r_H$ for every hyperplane H. The group is not abelian because rotations do not commute with reflections in general. ∎

2. *Prove that the elements e, a form a group (abelian) if the group operation is defined by*

$$ee = e, \quad ea = a, \quad ae = a, \quad aa = e.$$

Proof. By inspection, we see that e is the identity element, and both elements are their own inverses. Associativity may be checked by hand, examining all eight possible triples. And finally, the group is abelian because $ea = ae = a$. ∎

3. *Construct a multiplication table for the group of all permutations on three digits.*

Solution. For brevity, cycle notation is employed:

·	()	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
()	()	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
(1 2)	(1 2)	()	(1 3 2)	(1 2 3)	(2 3)	(1 3)
(1 3)	(1 3)	(1 2 3)	()	(1 3 2)	(1 2)	(2 3)
(2 3)	(2 3)	(1 3 2)	(1 2 3)	()	(1 3)	(1 2)
(1 2 3)	(1 2 3)	(1 3)	(2 3)	(1 2)	(1 3 2)	()
(1 3 2)	(1 3 2)	(2 3)	(1 2)	(1 3)	()	(1 2 3)

∎